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Spiral bond animals—ratio approach

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Abstract. Bond animals with a constraint of a given winding direction, on the square lattice, are enumerated up to 14 bonds. Numerical evidence further confirms our previous conjecture using the position space renormalisation group approach, that they belong to a new universality class.

1. Introduction

Recently there has been considerable interest in the study of various models simulating polymers, particularly the study of their critical behaviours using these models.

One of the most important and attractive subjects is the role of the macroscopic symmetry of the modelled system in the aggregation phenomena.

Now it is well accepted that the dimensionality, the number of interior degrees of freedom as well as the macroscopic symmetry of the system determine simultaneously the critical behaviours. As for the last factor, one can list examples such as the difference between the critical phenomenon taking place on the surface or interface and the corresponding one in the bulk (see e.g. a review by Binder 1983); the novel behaviour of aggregation happening on the directed lattice etc (e.g. directed percolation (Redner 1982), directed lattice animal (see e.g. Redner and Yang 1982, Dhar *et al* 1982, Dhar 1982) and the directed SAW (Redner and Majid 1983)).

It is obvious that the study of the system with reduced macroscopic symmetry is generally more difficult, but also rather peculiar and fascinating.

In this paper bond animals with a novel macroscopic symmetry, the spiral bond animal, proposed and studied by one of the present authors (Li 1984) using the position space renormalisation group (PSRG) approach, are exactly enumerated up to 14 bonds. Our exponent estimates further confirm the previous conjecture of a new universality class for the spiral bond animal (Li 1984) after comparing the exponents with those for unrestricted lattice animals.

2. Series expansion of the generating function, and the analysis of the mean-square size

In a previous paper (Li 1984) we defined a spiral bond animal as follows: A set of bonds $\mathscr{A}$ such that each bond $a \in \mathscr{A}$ is passable by and only by at least one spiral path (say clockwise path or paths) starting from a given fixed point (the origin) such that all bonds lying on the path (or paths) are in $\mathscr{A}$. (An example of a spiral bond animal is given in figure 1 illustrating this definition.) Briefly, a spiral bond animal is a

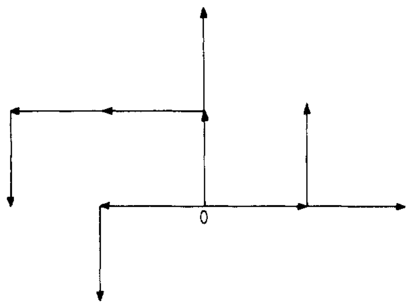

Figure 1. An example of spiral bond animals (with ten bonds).

connected bond cluster with a given winding direction constraint. Note that there exists a special site, the rotational centre, similar to the starting point of the directed lattice animal, while every site in the isotropic lattice animal is equivalent. The generating function $G(x)$ is defined in the usual way as the sum of the weights of all spiral bond animals; the weight of an animal of size N being X^N

$$G(x)=\sum_{\mathscr{A}} X^s=\sum_N C_N X^N \tag{1}$$

where C_N is the number of spiral animals of size N. Here we confine ourselves to the study of the 'random spiral animal' limit, as well as to the case without loops, i.e. spiral trees, for simplicity.

The starting point for the series expansion is the exact enumeration for C_N, the number of N-bond spiral animals on the lattice. We assume the following conventional asymptotic relations for large N for the spiral bond animal

$$C_N \sim N^{-\theta}\lambda^N, \tag{2}$$

$$\rho_N \sim N^{2\nu}, \tag{3}$$

where $C_N=\Sigma_{\{w_N\}} 1$ and $\rho_N=\Sigma_{\{w_N\}}\rho_{w_N}/\Sigma_{\{w_N\}} 1$, $\{w_N\}$ and ρ_{w_N} are the distinguished N-bond animals and square end-to-end distance respectively. We then define the growth parameter series through the successive ratio

$$\lambda_N \equiv C_N/C_{N-1}, \tag{4}$$

and the correlation-length exponent series

$$\nu_N \equiv \tfrac{1}{2}N(\rho_{N+1}/\rho_N-1)\sim \nu(1+\mathrm{O}(N^{-1})). \tag{5}$$

In table 1, we give the C_N's, $C_N\rho_N$'s and their corresponding successive ratios λ_N's and ν_N's for the square lattice up to $N=14$ (also shown in figures 2 and 3).

In addition to the previous ratio method, which was proposed by the present authors and has been extensively and successfully applied to 'trials' (Zhou and Li 1984a) and some other lattice models (Li and Zhou 1984b) an alternative method based on Stoltz's theorem (see, for example, Hobson 1926) is also used to study spiral bond animals. Stolz's theorem is as follows.

If the following conditions are fulfilled for the two sequences $\{X_N\}$ and $\{Y_N\}$:

$$Y_{N+1}>Y_N; \qquad \lim_{N\to\infty} X_N=+\infty \quad\text{and}\quad \lim_{N\to\infty} Y_N=+\infty,$$

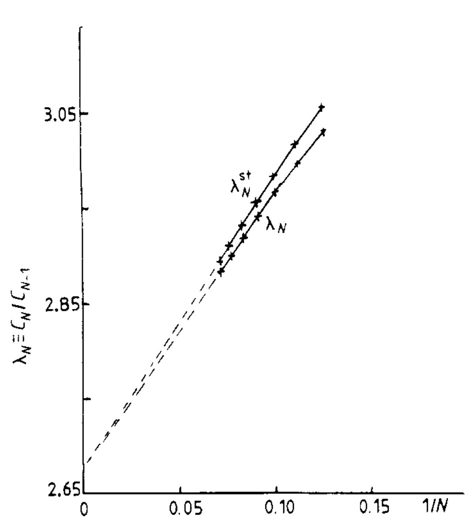

Figure 2. C_N/C_{N-1} against $1/N$ plot for square lattice, the superscript 'st' denotes the results obtained using the average method on the basis of Stolz's theorem.

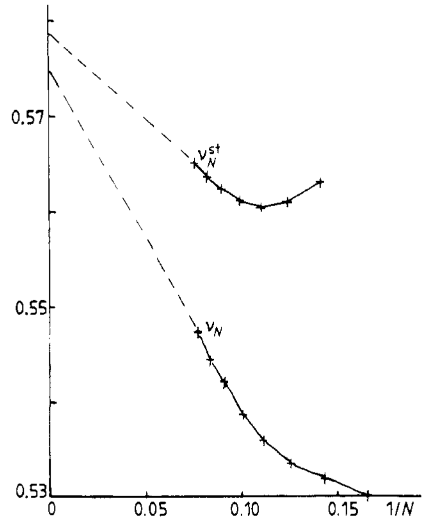

Figure 3. The successive ratios for the correlation-length exponents ν as a function of $1/N$ for a square lattice.

then we have $\lim_{N\to\infty} X_N/Y_N = A$, if the following limit exists

$$\lim_{N\to\infty} (X_N - X_{N-1})/(Y_N - Y_{N-1}) = A.$$

We define the sequences $\{X_N\}$ and $\{Y_N\}$ by

$$X_N = \sum_{m=0}^{N} C_m, \qquad Y_N = \sum_{m=0}^{N-1} C_m, \qquad C_0 = 1$$

(for which the conditions of Stolz's theorem are obviously fulfilled), where $C_m = \Sigma_{\{w_m\}} 1$, $\{w_m\}$ the distinguished m-bond configurations obtained by exact enumeration in the lattice model; then we have $\lim_{N\to\infty} X_N/Y_N \equiv \lim_{N\to\infty} \lambda_N^{st1} = \lambda$, if the following limit exists: $\lim_{N\to\infty} C_N/C_{N-1} \equiv \lim_{N\to\infty} \lambda_N = \lambda$, according to Stolz's theorem. Not only has one the equality of the limits for the sequences $\{\lambda_N\}$ and $\{\lambda_N^{st1}\}$, but we can also show that they have the same asymptotic behaviours (Zhou and Li 1984a) if one assumes some conventional behaviours asymptotically like $C_N \equiv \Sigma_{\{w_N\}} 1 \sim \lambda^N N^{-\theta}$ and $\rho_N \equiv \Sigma_{\{w_N\}} \rho_{w_N}/\Sigma_{\{w_N\}} 1 \sim AN^{2\nu}$, in which ρ_{w_N} is the square end-to-end distance; λ, θ and ν are respectively the corresponding location and exponents describing the nature of singularity in the related lattice model. Thus one has

$$\lambda_N^{st1} \equiv X_N/X_{N-1} = \lambda(1 - \theta/N + \mathrm{O}(1/N^2)) \tag{6$'$}$$

$$\nu_N^{st1} \equiv \frac{N}{2}\{[(\Sigma_{m=0}^{N+1} C_m\rho_m/X_{N+1})/(\Sigma_{m=0}^{N} C_m\rho_m/X_N)] - 1\} = \nu(1 + \mathrm{O}(1/N)) \tag{5$'$}$$

and

$$\nu_N^{st2} \equiv \frac{N}{2}\left(\sum_{m=1}^{N+1} \rho_m \Big/ \sum_{m=1}^{N} \rho_m - 1\right) - \frac{1}{2} = \nu(1 + \mathrm{O}(1/N)). \tag{5$''$}$$

The expressions for the previous ratio method are (5), and

$$\lambda_N \equiv C_N/C_{N-1} = \lambda(1-\theta/N+O(1/N^2)) \tag{6}$$

as well as

$$\theta_N = N(1-\lambda_N/\lambda') \tag{7}$$

according to (2) and (4), where λ' is the estimated value of the λ's. We have applied extensively this extended method to some typical lattice models (Zhou and Li 1984a, b, Li and Zhou 1984). We find that almost in every case studied, the results are comparable to (or often better than) those obtained by the previous ratio method. The sequence provided in the Neville table by the extended ratio method often appears to converge more rapidly and more steadily. One can even extend the Neville table to higher order.

The results based on Stolz's theorem are listed in the last two columns in table 1 and marked by the superscript 'st'.

From figure 2, we find the reasonably convergent results for λ_N's and λ_N^{stl}'s as follows

$$\lambda \sim 2.67 \pm 0.01, \tag{8}$$

$$\lambda^{\mathrm{stl}} \sim 2.67 \pm 0.01. \tag{9}$$

The linear projections are used to refine the results for the λ_N's and ν_N's. The linear projections and their means are defined as follows

$$X(n,m;e) = 1/(n-m)((n+e)X_n-(m+e)X_m) \qquad 0 \leqslant e \leqslant 0.5, \tag{10}$$

$$\bar{X}(n,m;e) = (X(n,m;e)+X(n-1,m-1;e))/2 \qquad 0 \leqslant e \leqslant 0.5. \tag{11}$$

Their values are listed in table 2. From table 2, we get the growth parameter

$$\lambda \sim 2.662 \pm 0.006, \tag{12}$$

$$\nu \sim 0.577 \pm 0.01, \tag{13}$$

$$-\theta \sim 1.19 \pm 0.03, \tag{14}$$

Table 1. The spiral bond animal problem on square lattice, C_N and ρ_N are the number and the mean-square end-to-end distance of N-bond spiral bond animals respectively.

N	C_N	$C_N\rho_N$	$\lambda_N(\equiv C_N/C_{N-1})$	ν_N	λ_N^{stl}	ν_N^{stl}
1	4	4				
2	14	30	3.5	0.6139	4.5	0.6043
3	48	166	3.428 57	0.5195	3.6667	0.5666
4	157	731	3.270 83	0.5296	3.3788	0.5692
5	504	2968	3.210 19	0.5375	3.2601	0.5714
6	1574	11 262	3.123 02	0.5335	3.1621	0.5677
7	4848	40 856	3.080 05	0.5320	3.1069	0.5628
8	14 698	142 694	3.031 76	0.5335	3.0557	0.5610
9	44 060	484 804	2.997 69	0.5359	3.0167	0.5605
10	130 732	1609 776	2.967 14	0.5386	2.9836	0.5610
11	384 620	5246 222	2.942 05	0.5421	2.9560	0.5625
12	1122 874	16 825 491	2.919 44	0.5442	2.9318	0.5636
13	3257 368	53 232 994	2.900 74	0.5471	2.9113	0.5652
14	9392 764	166 430 904	2.883 72		2.8932	

Table 2. The mean values (see (11)) of the linear projections of λ_N's and ν_N's, marked by $\bar{\lambda}'_N$ and $\bar{\nu}'_N$ respectively. The last four columns are the critical exponents θ_N for two testing λ's, where $-\theta_N = N(\lambda_N/\lambda' - 1)$.

N	$\bar{\lambda}'_N$	$\bar{\lambda}'^{\mathrm{st1}}_N$	$\bar{\nu}'_N$	$\bar{\nu}'^{\mathrm{st1}}_N$	$-\theta_N(\lambda'=2.662)$	$-\theta_N(\lambda'=2.666)$	$-\theta^{\mathrm{st1}}_N(\lambda'=2.662)$	$-\theta^{\mathrm{st1}}_N(\lambda'=2.666)$
7	2.7547	2.7240			1.0993	1.0872	1.1675	1.1553
8	2.7580	2.7286	0.5183	0.5413	1.1113	1.0976	1.1831	1.1693
9	2.7094	2.7012	0.5336	0.5404	1.1349	1.1197	1.1991	1.1838
10	2.7086	2.6942	0.5494	0.5521	1.1463	1.1296	1.2079	1.1911
11	2.6917	2.6825	0.5591	0.5617	1.1572	1.1390	1.2149	1.1966
12	2.6810	2.6729	0.5699	0.5719	1.1605	1.1408	1.2162	1.1964
13	2.6736	2.6658	0.5720	0.5766	1.1659	1.1446	1.2176	1.1963
14	2.6694	2.6617	0.5752	0.5801	1.1661	1.4333	1.2160	1.1932

and the fractal dimensionality

$$d_f = 1/\nu \sim 1.70. \tag{15}$$

Also the seuqences of Neville table of λ's and ν's for the spiral bond animals are respectively the following

$\lambda_0^{\mathrm{st1}} = 2.8932,$ $\lambda_1^{\mathrm{st1}} = 2.6617,$ $\lambda_2^{\mathrm{st1}} = 2.6337,$ $\lambda_3^{\mathrm{st1}} = 2.6381,$

$\lambda_4^{\mathrm{st1}} = 2.6528,$ $\lambda_5^{\mathrm{st1}} = 2.6610,$ $\lambda_6^{\mathrm{st1}} = 2.6641,$ $\lambda_7^{\mathrm{st1}} = 2.6641,$

$\nu_0^{\mathrm{st2}} = 0.5402,$ $\nu_1^{\mathrm{st2}} = 0.5522,$ $\nu_2^{\mathrm{st2}} = 0.5723,$ $\nu_3^{\mathrm{st2}} = 0.5836,$

$\nu_4^{\mathrm{st2}} = 0.5851,$ $\nu_5^{\mathrm{st2}} = 0.5823,$ $\nu_6^{\mathrm{st2}} = 0.5809,$ $\nu_7^{\mathrm{st2}} = 0.5819$

and $\nu_8^{\mathrm{st2}} = 0.5804$. The above sequences in the Neville table are for $N = 14$, and the subscripts denote the order in the Neville table.

Since the microscopic constraint causes a new macroscopic symmetry, which is essentially different from those either for the isotropic lattice or the directed one, thus one has reason to expect the possibilty of a new universality class for the spiral bond animal.

When the functionality $f = 2$, the allowed maximum number of bonds at any linked point, the spiral bond animal reduces to the spiral SAWs; this was proposed by Privman (1983), and has attracted much attention very recently (Blöte and Hilhorst 1984, Whittington 1984, Redner and de Arcanagelis 1984, Klein *et al* 1984, Guttmann and Wormald 1984). According to their results, the asymptotic form for the number of spiral SAWs is now known rigorously, and a new universality class as well as an essential singularity have been identified.

Since our reasonably convergent λ is far from 1, thus one can exclude the possibility with the singular form of $C_N \sim (\text{constant})^{\sqrt{N}}$, suggested by Redner and de Arcanagelis (1984) and has been proved and improved on a sound basis in the very recent works mentioned above ($C_N \sim 2^{-2}3^{-5/4}\pi N^{-7/4} \exp(2\pi(\frac{1}{3}N)^{1/2})$) for spiral SAWs.

There has been a rather wide range of estimates of ν for the lattice animal (for a review, see Stanley *et al* 1982), from 0.61–0.65. Perhaps the value of 0.6408 ± 0.003 by the phenomenological RG approach (Derrida and de Seze 1982) is the preferenble one. Our estimate of ν is well away from those for the usual 2D lattice animal and the spiral SAW. Although the estimate of θ here is rather crude, however, it is worth noticing that the θ for the spiral bond animal has an opposite sign compared with that for the unrestricted 2D lattice animals. This essentially different behaviour comes from the opposite sign of slopes in the $\lambda_N - 1/N$ plots for the restricted and unrestricted lattice animals, and is independent of any detailed numerical estimations for λ's. We think the above facts further confirm our previous conjecture of the new universality class for the spiral bond animal (Li 1984).

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